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Journal of Geometry and Physics 53 (2005) 249–258

JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Contact manifolds and generalized complex structures

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Received 7 May 2004; accepted 11 June 2004
Available online 27 July 2004

Abstract

We give simple characterizations of contact 1-forms in terms of Dirac structures. We also relate normal almost contact structures to the theory of Dirac structures.

MSC: 53DXX; 17B62; 17B63; 17B66

Keywords: Courant bracket; Dirac structure; Contact and Jacobi manifold

1. Introduction

Dirac structures on manifolds provide a unifying framework for the study of many geometric structures such as Poisson structures and closed 2-forms. They have applications to modeling of mechanical and electrical systems (see, for instance, [2]). Dirac structures were introduced by Courant and Weinstein (see [3,4]). Later, the theory of Dirac structures and Courant algebroids was developed in [11].

In [7], Hitchin defined the notion of a generalized complex structure on an even-dimensional manifold *M*, extending the setting of Dirac structures to the complex vector

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[1] Research partially supported by MCYT grant BFM 2003-01319.

doi:10.1016/j.geomphys.2004.06.006

bundle $(TM \oplus T^*M) \otimes \mathbb{C}$. This allows to include other geometric structures such as Calabi-Yau structures in the theory of Dirac structures. Furthermore, one gets a new way to look at Kähler structures (see [6]). However, the odd-dimensional analogue of the concept of a generalized complex structure was still missing. The aim of this Note is to fill this gap.

The first part of this paper concerns characterizations of contact 1-forms using the notion of an $\mathcal{E}^1(M)$-Dirac structure as introduced in [12]. In the second part, we define and study the odd-dimensional analogue of a generalized complex structure, which includes the class of almost contact structures. There are many distinguished subclasses of almost contact structures: contact metric, Sasakian, K-contact structures, etc. We hope that the theory of Dirac structures will lead to new insights on these structures.

2. $\mathcal{E}^1(M)$-Dirac structures

2.1. Definition and examples

In this section, we recall the description of several geometric structures (e.g. contact structures) in terms of Dirac structures.

First of all, observe that there is a natural bilinear form $\langle\cdot,\cdot\rangle$ on the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ defined by:

$$\langle (X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)\rangle = \tfrac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2 + f_1 g_2 + f_2 g_1)$$

for any $(X_j, f_j) + (\alpha_j, g_j) \in \Gamma(\mathcal{E}^1(M))$, with $j = 1, 2$. Moreover, for any integer $k \geq 1$, one can define

$$\tilde{d} : \Omega^k(M) \times \Omega^{k-1}(M) \to \Omega^{k+1}(M) \times \Omega^k(M),$$

by the formula

$$\tilde{d}(\alpha, \beta) = (\mathrm{d}\alpha,\ \alpha - \mathrm{d}\beta)$$

for any $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k-1}(M)$, where d is the exterior differentiation operator. When $k = 0$, we define $\tilde{d}f = (\mathrm{d}f, f)$. Clearly, $\tilde{d}^2 = 0$. We also have the contraction map given by:

$$i_{(X,f)}(\alpha, \beta) = (i_X\alpha + f\beta,\ -i_X\beta)$$

for any $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k-1}(M)$. From these two operations, we get

$$\tilde{\mathcal{L}}_{(X,f)} = i_{(X,f)} \circ \tilde{d} + \tilde{d} \circ i_{(X,f)}.$$

On the space of smooth sections of $\mathcal{E}^1(M)$, we define an operation similar to the Courant bracket by setting

$$
\begin{aligned}
&[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] \\
&\quad = ([X_1, X_2], X_1 \cdot f_2 - X_2 \cdot f_1) + \tilde{\mathcal{L}}_{(X_1, f_1)}(\alpha_2, g_2) - i_{(X_2, f_2)}\tilde{d}(\alpha_1, g_1) \qquad (1)
\end{aligned}
$$

for any $(X_j, f_j) + (\alpha_j, g_j) \in \Gamma(\mathcal{E}^1(M))$ with $j = 1, 2$. The skew-symmetric version of $[\cdot, \cdot]$ was introduced in [12]. One can notice that $\tilde{d}$ is nothing but the operator $d^{(0,1)}$ introduced [8]. Moreover, $\mathcal{E}^1(M)$ is an example of the so-called Courant-Jacobi algebroid (see [5]).

Definition 2.1 (Wade [12]). An $\mathcal{E}^1(M)$-Dirac structure is a sub-bundle L of $\mathcal{E}^1(M)$ which is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$ and integrable, i.e., $\Gamma(L)$ is closed under the bracket $[\cdot, \cdot]$.

Now, we consider some examples of $\mathcal{E}^1(M)$-Dirac structures.

(i) *Jacobi structures*. A *Jacobi structure* on a manifold M is given by a pair (π, E) formed by a bivector field π and a vector field E such that [10]

$$[E, \pi]_s = 0, \quad [\pi, \pi]_s = 2E \wedge \pi,$$

where $[\ ,\ ]_s$ is the Schouten–Nijenhuis bracket on the space of multi-vector fields. A manifold endowed with a Jacobi structure is called a *Jacobi manifold*. When E is zero, we get a Poisson structure.

Let (π, E) be a pair consisting of a bivector field π and a vector field E on M. Define the bundle map $(\pi, E)^\sharp : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ by setting

$$(\pi, E)^\sharp(\alpha, g) = (\pi^\sharp(\alpha) + gE, -i_E\alpha),$$

where α is a 1-form and $g \in C^\infty(M)$. The graph $L_{(\pi,E)}$ of $(\pi, E)^\sharp$ is an $\mathcal{E}^1(M)$-Dirac structure if and only if (π, E) is a Jacobi structure [12].

(ii) *Differential 1-forms*. Any pair (ω, η) formed by a 2-form ω and a 1-form η determines a maximally isotropic sub-bundle $L_{(\omega,\eta)}$ of $\mathcal{E}^1(M)$ given by:

$$(L_{(\omega,\eta)})_x = \{(X, f)_x + (i_X\omega + f\eta, -i_X\eta)_x : \ X \in \mathfrak{X}(M), f \in C^\infty(M)\}.$$

Moreover, we have that $\Gamma(L_{(\omega,\eta)})$ is closed under the bracket given by (1) if and only if $\omega = \mathrm{d}\eta$. The $\mathcal{E}^1(M)$-Dirac structure associated with a 1-form η will be denoted by L_η (see [9]).

2.2. Characterization of contact structures

In this section, we will characterize contact structures in terms of Dirac structures.

Let M be a $(2n+1)$-dimensional smooth manifold. A 1-form η on M is *contact* if $\eta \wedge (\mathrm{d}\eta)^n \neq 0$ at every point. There arises the question of how this condition translates into properties for L_η.

First, we give a characterization of Dirac structures coming from Jacobi structures (respectively, from differential 1-forms).

Proposition 2.2. *A sub-bundle L of $\mathcal{E}^1(M)$ is of the form $L_{(\Lambda,E)}$ (resp., $L_{(\omega,\eta)}$) for a pair $(\Lambda, E) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$ (resp., $(\omega, \eta) \in \Omega^2(M) \times \Omega^1(M)$) if and only if*

(i) *L is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$.*
(ii) *$L_x \cap ((T_xM \times \mathbb{R}) \oplus \{0\}) = \{0\}$ (resp., $L_x \cap (\{0\} \oplus (T_x^*M \times \mathbb{R})) = \{0\}$) for every $x \in M$.*

Moreover, (Λ, E) is a Jacobi structure, (resp. $\omega = \mathrm{d}\eta$) if and only if $\Gamma(L)$ is closed under the extended Courant bracket (1).

Proof. The proof of this proposition is straightforward (see [4] for the linear case). It is left to the reader. □

Now, let η be a contact structure on M. Then there exists an isomorphism $\flat_\eta : \mathfrak{X}(M) \to \Omega^1(M)$ given by $\flat_\eta(X) = i_X \,\mathrm{d}\eta + \eta(X)\eta$ which allows us to construct a Jacobi structure (π, E) given by:

$$\pi(\alpha, \beta) = \mathrm{d}\eta(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)) \quad \text{for } \alpha, \beta \in \Omega^1(M), \qquad E = \flat_\eta^{-1}(\eta),$$

which satisfies that $((\pi, E)^\sharp)^{-1}(X, f) = (-i_X \,\mathrm{d}\eta - f\eta, \eta(X))$. Moreover, if (π, E) is a Jacobi structure such that $(\pi, E)^\sharp$ is an isomorphism then it comes from a contact structure. From these facts, we deduce that for a contact structure $L_\eta \cong L_{(\pi,E)}$. As a consequence of this result and Proposition 2.2, one gets:

Theorem 2.3. *There is a one-to-one correspondence between contact* 1*-forms on a* $(2n+1)$*-dimensional manifold and* $\mathcal{E}^1(M)$*-Dirac structures satisfying the properties*

$$L_x \cap ((T_xM \times \mathbb{R}) \oplus \{0\}) = \{0\},$$

$$L_x \cap (\{0\} \oplus (T_x^*M \times \mathbb{R})) = \{0\}$$

for every $x \in M$.

Another characterization is the following:

Theorem 2.4. *An $\mathcal{E}^1(M)$-Dirac structure L_η corresponds to a contact* 1*-form η if and only if*

$$L_\eta \cap ((TM \times \{0\}) \oplus (\{0\} \times \mathbb{R})),$$

is a 1*-dimensional sub-bundle of $\mathcal{E}^1(M)$ generated by an element of the form* $(\xi, 0) + (0, -1)$.

Proof. Indeed, if $e_X = (X, 0) + (0, -i_X\eta)$ then $e_X \in L_\eta$ if and only if

$$\langle (Y, g) + (i_Y\, \mathrm{d}\eta + g\eta, -i_Y\eta), e_X\rangle = 0, \quad \forall (Y, g) \in \mathfrak{X}(M) \times C^\infty(M),$$

but this is equivalent to $\mathrm{d}\eta(X, Y) = 0$, for all $Y \in \mathfrak{X}(M)$.

This shows $L_\eta \cap ((TM \times \{0\}) \oplus (\{0\} \times \mathbb{R}))$ is a one-dimensional sub-bundle of $\mathcal{E}^1(M)$ if and only if Ker $\mathrm{d}\eta$ is a one-dimensional sub-bundle of TM. If $(\xi, 0) + (0, -1)$ generates $L_\eta \cap (TM \times \{0\} \oplus \{0\} \times \mathbb{R})$ then

$$\langle (\xi, 0) + (0, -1), (0, 1) + (\eta, 0)\rangle = \eta(\xi) - 1 = 0.$$

Therefore,

$$\mathrm{Ker}\, \mathrm{d}\eta \cap \mathrm{Ker}\, \eta = \{0\}.$$

We conclude that η is a contact form. Moreover ξ is nothing but the corresponding Reeb field, i.e., the vector field characterized by the equations $i_\xi\, \mathrm{d}\eta = 0$ and $\eta(\xi) = 1$. The converse is obvious. □

3. Generalized complex structures

In this section, we will recall the notion of generalized complex structures.

Definition 3.1 (Gualtieri [6]). Let M be a smooth even-dimensional manifold. A *generalized almost complex structure* on M is a sub-bundle E of the complexification $(TM \oplus T^*M) \otimes \mathbb{C}$ such that

(i) E is isotropic.
(ii) $(TM \oplus T^*M) \otimes \mathbb{C} = E \oplus \bar{E}$, where $\bar{E}$ is the conjugate of E.

The terminology is justified by the following result:

Proposition 3.2 (Gualtieri [6]). *There is a one-to-one correspondence between generalized almost complex structures and endomorphisms $\mathcal{J}$ of the vector bundle $TM \oplus T^*M$ such that $\mathcal{J}^2 = -\mathrm{id}$ and $\mathcal{J}$ is orthogonal with respect to $\langle\cdot, \cdot\rangle$.*

Proof. Suppose that E is a generalized almost complex structure on M. Define

$$\mathcal{J}(e) = \sqrt{-1}e, \qquad \mathcal{J}(\bar{e}) = -\sqrt{-1}\bar{e} \quad \text{for any } e \in \Gamma(E).$$

Then, $\mathcal{J}$ satisfies the properties $\mathcal{J}^2 = -\mathrm{i}d$ and $\mathcal{J}^* = -\mathcal{J}$. Conversely, assume that $\mathcal{J}$ satisfies these two properties. Define the sub-bundle E whose fibre is the $\sqrt{-1}$-eigenspace of $\mathcal{J}$. It is not difficult to prove that E is isotropic under $\langle\cdot, \cdot\rangle$. Moreover, since $\bar{E}$ is just the $(-\sqrt{-1})$-eigenspace of $\mathcal{J}$ we get that $(TM \oplus T^*M) \otimes \mathbb{C} = E \oplus \bar{E}$. □

We have the following definition:

Definition 3.3. Let M be an even-dimensional smooth manifold. A generalized almost complex structure $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ is *integrable* if it is closed under the Courant bracket. Such a sub-bundle is called a *generalized complex structure*.

The notion of a generalized complex structure on an even-dimensional smooth manifold was introduced by Hitchin in [7].

4. Generalized almost contact structures

The existence of a generalized almost complex structure on M forces the dimension of M to be even (see [6]). A natural question to ask is: what would be the odd-dimensional analogue of a generalized almost complex structure?

To define the analogue of the concept of a generalized almost complex structure for odd-dimensional manifolds, one should consider the vector bundle $\mathcal{E}^1(M) \otimes \mathbb{C}$ instead of $(TM \oplus T^*M) \otimes \mathbb{C}$.

Definition 4.1. Let M be a real smooth manifold of dimension $d = 2n + 1$. A *generalized almost contact structure* on M is a sub-bundle E of $\mathcal{E}^1(M) \otimes \mathbb{C}$ such that E is isotropic and

$$\mathcal{E}^1(M) \otimes \mathbb{C} = E \oplus \overline{E},$$

where $\overline{E}$ is the complex conjugate of E.

By a proof similar to that of Proposition 3.2, one gets the following result.

Proposition 4.2. *Let M be a real smooth manifold of dimension $d = 2n + 1$. There is a one-to-one correspondence between generalized almost contact structures on M and endomorphisms $\mathcal{J}$ of the vector bundle $\mathcal{E}^1(M)$ such that $\mathcal{J}^2 = -\mathrm{i}d$ and $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.*

4.1. Examples

4.1.1. Almost contact structures

Let M be a smooth manifold of dimension $d = 2n + 1$. An *almost contact structure* on M is a triple (φ, ξ, η), where φ is a (1,1)-tensor field, ξ is a vector field on M, and η is a 1-form such that

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2(X) = -X + \eta(X)\xi, \quad \forall X \in \mathfrak{X}(M),$$

(see [1]). As a first consequence, we get that

$$\varphi(\xi) = 0, \qquad \eta \circ \varphi = 0.$$

We now show that every almost contact structure determines a generalized almost contact structure. Define $J : \Gamma(TM \times \mathbb{R}) \to \Gamma(TM \times \mathbb{R})$ by:

$$J(X, f) = (\varphi X - f\xi, \eta(X)) \quad \text{for all } X \in \mathfrak{X}(M),\ f \in C^\infty(M).$$

Then $J^2 = -id$. Let J^* be the dual map of J. Consider the endomorphism $\mathcal{J}$ defined by:

$$\mathcal{J}(u) = J(X, f) - J^*(\alpha, g)$$

for $u = (X, f) + (\alpha, g) \in \Gamma(\mathcal{E}^1(M))$. Then $\mathcal{J}$ satisfies $\mathcal{J}^2 = -id$ and $\mathcal{J}^* = -\mathcal{J}$.

In addition, one can deduce that the generalized almost contact structure E is given by:

$$E = F \oplus \text{Ann}(F), \tag{2}$$

where

$$F_x = \{J(X, f)_x + \sqrt{-1}(X, f)_x | (X, f) \in \Gamma(TM \times \mathbb{R})\}, \tag{3}$$

and $\text{Ann}(F)$ is the annihilator of E.

4.1.2. Almost cosymplectic structures

An almost cosymplectic structure on a smooth manifold M of dimension $d = 2n + 1$ is a pair (ω, η) formed by a 2-form ω and a 1-form η such that $\eta \wedge \omega^n \neq 0$ everywhere. The map $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by:

$$\flat(X) = i_X\omega + \eta(X)\eta, \quad \forall X \in \mathfrak{X}(M),$$

is an isomorphism of $C^\infty(M)$-modules. The vector field $\xi = \flat^{-1}(\eta)$ is called the Reeb vector field of the almost cosymplectic structure and it is characterized by $i_\xi\omega = 0$ and $\eta(\xi) = 1$. Define $\Theta : \mathfrak{X}(M) \times C^\infty(M) \to \Omega^1(M) \times C^\infty(M)$ by:

$$\Theta(X, f) = (i_X\omega + f\eta, -\eta(X)), \quad \forall X \in \mathfrak{X}(M),\ \forall f \in C^\infty(M).$$

One can check that Θ is an isomorphism of $C^\infty(M)$-modules. Let $\mathcal{J} : \Gamma(\mathcal{E}^1(M)) \to \Gamma(\mathcal{E}^1(M))$ be the endomorphism given by:

$$\mathcal{J}((X, f) + (\alpha, g)) = -\Theta^{-1}(\alpha, g) + \Theta(X, f).$$

It is easy to check that $\mathcal{J}^2 = -id$. Moreover, for $e_i = (X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))$, we have

$$\langle \mathcal{J}e_1, e_2 \rangle = \langle -\Theta^{-1}(\alpha_1, g_1) + \Theta(X_1, f_1), (X_2, f_2) + (\alpha_2, g_2) \rangle = -\langle e_1, \mathcal{J}e_2 \rangle.$$

Hence $\mathcal{J}^* = -\mathcal{J}$.

This shows that every almost cosymplectic structure determines a generalized almost contact structure. Furthermore, the associated bundle E is given by:

$$E_x = \{(X, f)_x - \sqrt{-1}\Theta(X, f)_x | (X, f) \in \Gamma(TM \times \mathbb{R})\}. \tag{4}$$

5. Integrability

By analogy to generalized complex structures, one can consider the integrability of a generalized almost contact structure.

Definition 5.1. On an odd-dimensional smooth manifold M, we say that a generalized almost contact structure $E \subset \mathcal{E}^1(M) \otimes \mathbb{C}$ is *integrable* if it is closed under the extended Courant bracket given by Eq. (1).

5.1. Examples

5.1.1. Normal almost contact structures

An almost contact structure (φ, ξ, η) is *normal* if

$$N_\varphi(X, Y) + \mathrm{d}\eta(X, Y)\xi = 0 \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

where N_φ is the Nijenhuis torsion of φ, i.e.,

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

Some properties of normal almost contact structures are the following ones (see [1]).

Lemma 5.2. *If an almost contact structure* (φ, ξ, η) *is normal then it follows that*

$$\mathrm{d}\eta(X, \xi) = 0, \quad \eta[\varphi X, \xi] = 0, \quad [\varphi X, \xi] = \varphi[X, \xi], \quad \mathrm{d}\eta(\varphi X, Y) = \mathrm{d}\eta(\varphi Y, X)$$

for $X, Y \in \mathfrak{X}(M)$.

Proof. Applying normality condition to $Y = \xi$ we get that

$$0 = N_\varphi(X, \xi) + \mathrm{d}\eta(X, \xi)\xi = \varphi^2[X, \xi] - \varphi[\varphi X, \xi] + \mathrm{d}\eta(X, \xi)\xi.$$

Using the fact that $\eta \circ \varphi = 0$, we obtain $\mathrm{d}\eta(X, \xi) = 0$, for any $X \in \mathfrak{X}(M)$. As a consequence, $\eta[\varphi X, \xi] = 0$. On the other hand,

$$\begin{aligned} 0 &= N_\varphi(\varphi X, \xi) + \mathrm{d}\eta(\varphi X, \xi)\xi \\ &= \varphi^2[\varphi X, \xi] - \varphi[\varphi^2 X, \xi] + \mathrm{d}\eta(\varphi X, \xi)\xi = -[\varphi X, \xi] + \varphi[X, \xi]. \end{aligned}$$

Finally, if $X, Y \in \mathfrak{X}(M)$ then

$$\eta(N_\varphi(\varphi X, Y) + \mathrm{d}\eta(\varphi X, Y)\xi) = -\eta([\varphi^2 X, Y] + [\varphi X, \varphi Y]) + \mathrm{d}\eta(\varphi X, Y).$$

We deduce that $\mathrm{d}\eta(\varphi X, Y) = \mathrm{d}\eta(\varphi Y, X)$. □

We have seen that every almost contact structure (φ, ξ, η) determines a generalized almost complex structure $E \subset \mathcal{E}^1(M) \otimes \mathbb{C}$. Furthermore, we have the following result:

Theorem 5.3. *An almost contact structure* (φ, ξ, η) *is normal if and only if its corresponding sub-bundle E given by* (2) *is integrable.*

Proof. Clearly, the integrability of E is equivalent to the closedness of $\Gamma(F)$ under the extended Courant bracket, where F is the sub-bundle defined by (3). Suppose $[\Gamma(F), \Gamma(F)] \subset \Gamma(F)$. Let $u_X = (X, 0)$, $u_Y = (Y, 0) \in \Gamma(\mathcal{E}^1(M))$. Denote $e_X = Ju_X + \sqrt{-1}u_X$ and $e_Y = Ju_Y + \sqrt{-1}u_Y$. Then

$$[e_X, e_Y] \in F \Leftrightarrow [Ju_X, Ju_Y] - [u_X, u_Y] = J([Ju_X, u_Y] + [u_X, Ju_Y]).$$

By a simple computation, one gets

$$[Ju_X, Ju_Y] - [u_X, u_Y] = ([\varphi X, \varphi Y] - [X, Y], \varphi X \cdot \eta(Y) - \varphi Y \cdot \eta(X)).$$

Moreover, the term $J([Ju_X, u_Y] + [u_X, Ju_Y])$ equals

$$(\varphi([\varphi X, Y] + [X, \varphi Y]) - (X \cdot \eta(Y) - Y \cdot \eta(X))\xi, \eta([\varphi X, Y] + [X, \varphi Y])).$$

Therefore $[e_X, e_Y] \in \Gamma(F)$ if and only if

$$[\varphi X, \varphi Y] - [X, Y] = \varphi([\varphi X, Y] + [X, \varphi Y]) - (X \cdot \eta(Y) - Y \cdot \eta(X))\xi,$$
$$\varphi X \cdot \eta(Y) - \varphi Y \cdot \eta(X) = \eta([\varphi X, Y] + [X, \varphi Y]).$$

Because $[X, Y] = -\varphi^2([X, Y]) + \eta([X, Y])\xi$ and $\eta(\varphi X) = 0$, for any $X, Y \in \mathfrak{X}(M)$, this implies the relations

$$N_\varphi(X, Y) + \mathrm{d}\eta(X, Y)\xi = 0, \qquad \mathrm{d}\eta(\varphi X, Y) = \mathrm{d}\eta(\varphi Y, X).$$

This proves that if E is integrable then the almost contact structure is normal. Conversely, suppose that $N_\varphi(X, Y) + \mathrm{d}\eta(X, Y)\xi = 0$, for any X, Y in $\mathfrak{X}(M)$. Using Lemma 5.2, we also have that $\mathrm{d}\eta(\varphi X, Y) = \mathrm{d}\eta(\varphi Y, X)$. Thus, we conclude that $[e_X, e_Y] \in \Gamma(F)$, for any $e_X = u_X + \sqrt{-1}Ju_X$, $e_Y = u_Y + \sqrt{-1}Ju_Y$ in $\Gamma(F)$.

It remains to show that $[e_X, J(0, 1) + \sqrt{-1}(0, 1)]$ is in $\Gamma(F)$, for any section $e_X = Ju_X + \sqrt{-1}u_X \in \Gamma(F)$. This condition is equivalent to the relations

$$[\varphi X, \xi] = \varphi[X, \xi], \qquad \xi \cdot \eta(X) = -\eta([X, \xi]).$$

The relation $\xi \cdot \eta(X) = -\eta([X, \xi])$ is satisfied since $\mathrm{d}\eta(X, \xi) = 0$ by Lemma 5.2. We conclude that $[e_X, J(0, 1) + \sqrt{-1}(0, 1)] \in F$. Therefore F is closed under that extended Courant bracket, which means that E is integrable. □

5.1.2. Contact structures

Let (ω, η) be an almost cosymplectic structure and E the associated generalized almost contact structure given by (4). We will prove that the integrability condition forces η to be a contact structure. In fact,

Proposition 5.4. *Let (ω, η) be an almost cosymplectic structure on a manifold M and E the associated generalized almost contact structure. Then, E is integrable if and only if $\omega = \mathrm{d}\eta$. As a consequence, η is a contact structure on M.*

Proof. Let $e_1, e_2 \in \Gamma(E)$. One can easily show that $[e_1, e_2] \in \Gamma(E)$ if and only if $\omega = \mathrm{d}\eta$. □

Remark 5.5. Following [6], one can define an analogue of generalized Kähler structure. In our setting, one could define the notion of a generalized Sasakian structure as a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized integrable generalized almost contact structures, i.e. $\mathcal{J}_1 \circ \mathcal{J}_2 = \mathcal{J}_2 \circ \mathcal{J}_1$, such that $G = -\mathcal{J}_1\mathcal{J}_2$ defines a positive definite metric on $\mathcal{E}^1(M)$. In particular, every Sasakian structure is a generalized Sasakian structure. We postpone the study of this notion and its main properties to a separate paper.

Acknowledgments

D. Iglesias wishes to thank the Spanish Ministry of Education and Culture and Fulbright program for a MECD/Fulbright postdoctoral grant.

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